Free Taster Resource from Bradley's Maths

(I)GCSE Extended Level Mathematics (0580)

E7.4 Vector Geometry

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Abstract

Thank you for downloading this free taster resource from **Bradley's Maths**. I hope you and your students find it useful.

This worksheet is a sample from the comprehensive **E7 Transformations and Vectors, E8 Probability and E9 Statistics** booklets, which together contain 16 worksheets covering every aspect of this section of the Cambridge IGCSE (0580) syllabus.

Each full booklet comes with a companion Answer Booklet containing fully worked, exam-style model answers and explanations for every question. Each worksheet and answer sheet has a Key Concepts and Formulas section with methods, pro-tips, galleries, deeper insights, cautionary notes, and in the answer sheets, reminders. These have been written with the student in mind in order to assist them in fully understanding the mathematics

All resources are meticulously crafted and professionally typeset using LaTeX for exceptional clarity and quality.

You can find the full E7, E8 and E69 booklets and other resources by searching for "Bradley's Maths" on the TES website.

Worksheet Instructions

- Answer all questions in the spaces provided.
- Show all your working clearly, step-by-step.
- Express vectors in terms of the given bold letters (e.g., $\boldsymbol{a}$, $\boldsymbol{c}$).
- Simplify your vector expressions where possible.
- Calculators are not required for this worksheet.

Key Concepts: Vector Geometry

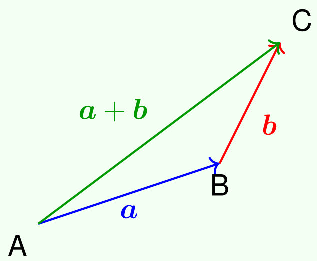
Deeper Insight: Vector Geometry is Pathfinding

Vector geometry is the art of describing journeys around a shape. Instead of using lengths and angles, we use vectors to define paths along the edges. Every problem can be solved by finding a route from your start point to your end point using the vectors you already know.

Method: The Fundamental Rules of Vector Journeys

1. The Triangle Law for Addition (Nose-to-Tail)
To add two vectors, you draw them one after the other. The resultant vector is the direct path from the very start to the very end.

$$\vec{AC} = \vec{AB} + \vec{BC}$$



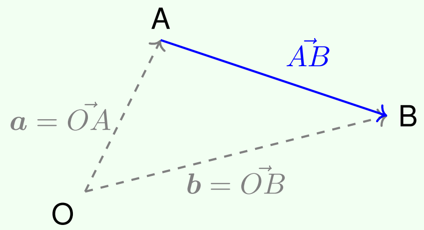
2. Negative Vectors (Reversing the Journey)
The vector $\vec{BA}$ is the exact opposite journey to $\vec{AB}$. It has the same length but the opposite direction.

$$\vec{BA} = -\vec{AB} \quad \text{or simply} \quad -\boldsymbol{a}$$

Method: Finding the Vector Between Two Points

The most common task is finding the vector journey between two points, A and B, using their position vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, which start from the origin O. The rule is "End - Start".

$$\vec{\boldsymbol{AB}} = \vec{\boldsymbol{OB}} - \vec{\boldsymbol{OA}} \quad \text{or simply} \quad \vec{\boldsymbol{AB}} = \boldsymbol{b} - \boldsymbol{a}$$



This comes from the path $\vec{AB} = \vec{AO} + \vec{OB}$. Since $\vec{AO}$ is the opposite of $\vec{OA}$, we can write this as:

$$\vec{AB} = -\vec{OA} + \vec{OB} = -\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} - \boldsymbol{a}$$

Worked Example: Finding a Vector from Coordinates

Example: Point A has position vector $\boldsymbol{a} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ and point B has position vector $\boldsymbol{b} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$. Find $\vec{AB}$.

$$\vec{AB} = \boldsymbol{b} - \boldsymbol{a}$$

$$= \begin{pmatrix} 6 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} 6 - 2 \\ 3 - 5 \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{4} \\ \mathbf{-2} \end{pmatrix}$$

Caution: Direction is Everything! ($\vec{AB}$ vs $\vec{BA}$)

The order of the letters defines the direction of the journey. $\vec{AB}$ (from A to B) is the opposite of $\vec{BA}$ (from B to A).

$$\vec{\boldsymbol{BA}} = \boldsymbol{a} - \boldsymbol{b} = -(\boldsymbol{b} - \boldsymbol{a}) = -\vec{\boldsymbol{AB}}$$

Strategy: Points Along a Line (Midpoints and Ratios)

To find the position vector of a point along a line segment AB, start at A and travel a fraction of the way along the vector $\vec{AB}$.

To find the midpoint M of AB:

$$\vec{OM} = \vec{OA} + \frac{1}{2}\vec{AB}$$

$$= \boldsymbol{a} + \frac{1}{2}(\boldsymbol{b} - \boldsymbol{a})$$

$$= \boldsymbol{a} + \frac{1}{2}\boldsymbol{b} - \frac{1}{2}\boldsymbol{a}$$

$$= \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{a} + \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{b} \quad \text{or} \quad \frac{\mathbf{1}}{\mathbf{2}}(\boldsymbol{a} + \boldsymbol{b})$$

To find a point X that divides AB in the ratio $m : n$:

$$\vec{OX} = \vec{OA} + \frac{m}{m+n}\vec{AB}$$

$$= \boldsymbol{a} + \frac{m}{m+n}(\boldsymbol{b} - \boldsymbol{a})$$

Strategy: Parallel Vectors and Collinear Points

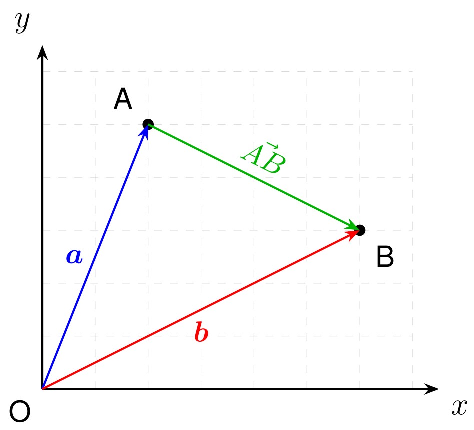
Two vectors are **parallel** if one is a scalar multiple of the other.

If $\boldsymbol{p} = k\boldsymbol{q}$ (where k is a number), then $\boldsymbol{p}$ is parallel to $\boldsymbol{q}$.

Collinear points are points that all lie on the same straight line. To prove that A, B, and C are collinear, you must show two things:

1. The vector $\vec{AB}$ is parallel to the vector $\vec{BC}$ (i.e., $\vec{AB} = k\vec{BC}$).
2. The two vectors share a common point (in this case, point B).

1. O is the origin. The position vector of point A is $\boldsymbol{a} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ and the position vector of point B is $\boldsymbol{b} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$. *Recall: The position vector of A, $\vec{OA}$, is often written as $\boldsymbol{a}$.*

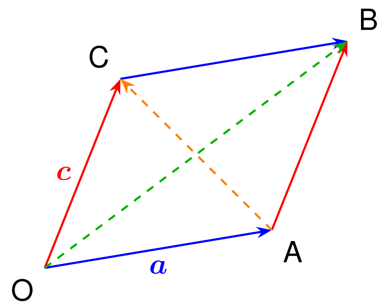


(a) Express the vector $\vec{AB}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$. [1] *(Hint: Use the path $\vec{AO} + \vec{OB}$)*

(b) Hence, find $\vec{AB}$ as a column vector. [2]

Total: [3]

2. $OABC$ is a parallelogram. $\vec{OA} = \boldsymbol{a}$ and $\vec{OC} = \boldsymbol{c}$.



Express the following vectors in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$:

(a) $\vec{OB}$ (the diagonal) [1]

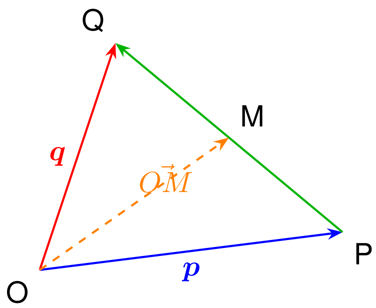
(b) $\vec{AC}$ (the other diagonal) [2]

(c) $\vec{BC}$ [1]

(d) $\vec{BA}$ [1]

Total: [5]

3. In triangle OPQ, $\vec{OP} = \boldsymbol{p}$ and $\vec{OQ} = \boldsymbol{q}$. M is the midpoint of PQ.



Express the following vectors in terms of $\boldsymbol{p}$ and $\boldsymbol{q}$. Simplify your answers.

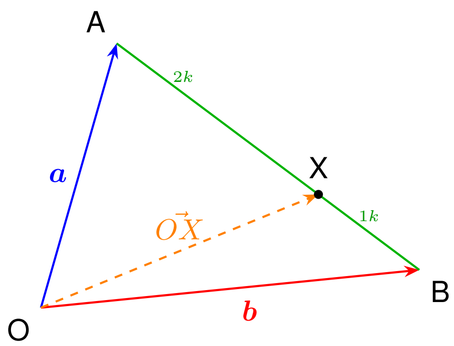
(a) $\vec{PQ}$ [1]

(b) $\vec{PM}$ [1]

(c) $\vec{OM}$ [2] *(Hint: Use the path* $\vec{OP} + \vec{PM}$ *or* $\vec{OQ} + \vec{QM}$*)*

Total: [4]

4. O is the origin. $\vec{OA} = \boldsymbol{a}$ and $\vec{OB} = \boldsymbol{b}$. Point X lies on the line AB such that $AX : XB = 2 : 1$.



Express the following vectors in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$. Simplify your answers.

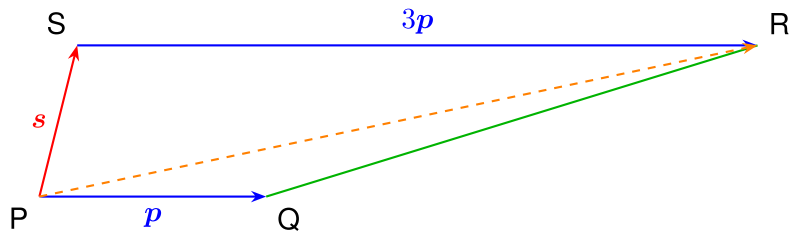
(a) $\vec{AB}$ [1]

(b) $\vec{AX}$ [1] *(Hint: It's a fraction of* $\vec{AB}$*)*

(c) $\vec{OX}$ [2] *(Hint: Use the path* $\vec{OA} + \vec{AX}$*)*

Total: [4]

5. $PQRS$ is a trapezium with PQ parallel to SR. $\vec{PQ} = \boldsymbol{p}$, $\vec{PS} = \boldsymbol{s}$, and $SR = 3PQ$. (Note: $\vec{SR}$ will be $3\boldsymbol{p}$ as $PQ \parallel SR$).



Express the following vectors in terms of $\boldsymbol{p}$ and $\boldsymbol{s}$. Simplify your answers.

(a) $\vec{PR}$ [2] *(Hint: Use path $PS + SR$)*

(b) $\vec{QS}$ [2] *(Hint: Use path $QP + PS$)*

(c) $\vec{QR}$ [2] *(Hint: Use path $QP + PS + SR$ or $QP + PR$)*

Total: [6]

6. Point L has coordinates $(1, 4)$, M has coordinates $(5, 1)$ and N has coordinates $(-2, 0)$. O is the origin.

   (a) Write down the position vectors $\vec{OL}$, $\vec{OM}$, and $\vec{ON}$ as column vectors. [3]
   $\vec{OL} = \vec{OM} = \vec{ON} =$

   (b) Express $\vec{LM}$ as a column vector. [2]

   (c) Express $\vec{MN}$ as a column vector. [2]

   (d) Express $\vec{NL}$ as a column vector. [2]

   (e) Show that $\vec{LM} + \vec{MN} + \vec{NL} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ (the zero vector). [2]

Total: [11]

This is the end of the worksheet

Worked Solutions

1) (a)

> **Reminder: Finding the Vector Between Two Points**
>
> To find the vector journey from A to B, $\vec{AB}$, the rule is always ”End point minus Start point”: $\vec{AB} = \vec{OB} - \vec{OA}$, or simply $\boldsymbol{b} - \boldsymbol{a}$.

The path from A to B can be thought of as going backwards along $\boldsymbol{a}$ to the origin $(-\boldsymbol{a})$, and then forwards along $\boldsymbol{b}$ to B.

$$\vec{AB} = \vec{AO} + \vec{OB}$$

$$= -\vec{OA} + \vec{OB}$$

$$= -\boldsymbol{a} + \boldsymbol{b} \quad \text{or} \quad \boldsymbol{b} - \boldsymbol{a}$$

(b) We now substitute the column vectors for $\boldsymbol{a}$ and $\boldsymbol{b}$.

$$\vec{AB} = \boldsymbol{b} - \boldsymbol{a}$$

$$= \begin{pmatrix} 6 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} 6-2 \\ 3-5 \end{pmatrix} = \begin{pmatrix} \mathbf{4} \\ \mathbf{-2} \end{pmatrix}$$

2)

Pro-Tip: Using Properties of a Parallelogram

In a parallelogram, opposite sides are parallel and equal in length. This means their vectors are identical. Here, $\vec{BC} = \vec{OA} = \boldsymbol{a}$ and $\vec{AB} = \vec{OC} = \boldsymbol{c}$.

(a) To find $\vec{OB}$, we can follow the path $\vec{OA} + \vec{AB}$.

$$\vec{OB} = \vec{OA} + \vec{AB}$$

$$= \boldsymbol{a} + \boldsymbol{c}$$

(b) To find $\vec{AC}$, we can follow the path $\vec{AO} + \vec{OC}$.

$$\vec{AC} = \vec{AO} + \vec{OC}$$

$$= -\boldsymbol{a} + \boldsymbol{c} \quad \text{or} \quad \boldsymbol{c} - \boldsymbol{a}$$

(c) From the properties of the parallelogram, $\vec{BC}$ is the same as $\vec{OA}$. So, $\vec{BC} = \boldsymbol{a}$.

(d) The vector $\vec{BA}$ is the opposite of $\vec{AB}$. Since $\vec{AB} = \boldsymbol{c}$, $\vec{BA} = -\boldsymbol{c}$.

3)

(a) To find $\vec{PQ}$, we use the rule "End - Start".

$$\vec{PQ} = \vec{OQ} - \vec{OP}$$

$$= \boldsymbol{q} - \boldsymbol{p}$$

(b) M is the midpoint of PQ, so the vector journey from P to M is half the journey from P to Q.

$$\vec{PM} = \frac{1}{2}\vec{PQ}$$

$$= \boldsymbol{\frac{1}{2}(q - p)}$$

(c) To find $\vec{OM}$, we follow the path from O to P, then from P to M.

$$\vec{OM} = \vec{OP} + \vec{PM}$$

$$= \boldsymbol{p} + \frac{1}{2}(\boldsymbol{q} - \boldsymbol{p})$$

$$= \boldsymbol{p} + \frac{1}{2}\boldsymbol{q} - \frac{1}{2}\boldsymbol{p}$$

$$= \boldsymbol{\frac{1}{2}p + \frac{1}{2}q} \quad \text{or} \quad \boldsymbol{\frac{1}{2}(p + q)}$$

Deeper Insight: The Midpoint Formula for Position Vectors

This result reveals a powerful general rule: the position vector of the midpoint of a line segment is the average (mean) of the position vectors of its endpoints.

4)

(a) To find $\vec{AB}$, use "End - Start".

$$\vec{AB} = \vec{OB} - \vec{OA}$$

$$= \boldsymbol{b} - \boldsymbol{a}$$

(b) The ratio $AX : XB = 2 : 1$ means the total line segment AB is divided into $2 + 1 = 3$ parts. The journey from A to X is $\frac{2}{3}$ of the total journey from A to B.

$$\vec{AX} = \frac{2}{3}\vec{AB}$$

$$= \boldsymbol{\frac{2}{3}(b - a)}$$

(c) To find $\vec{OX}$, we follow the path from O to A, then from A to X.

$$\vec{OX} = \vec{OA} + \vec{AX}$$

$$= \boldsymbol{a} + \frac{2}{3}(\boldsymbol{b} - \boldsymbol{a})$$

$$= \boldsymbol{a} + \frac{2}{3}\boldsymbol{b} - \frac{2}{3}\boldsymbol{a}$$

$$= \boldsymbol{\frac{1}{3}a + \frac{2}{3}b}$$

5)

Caution: Pay Attention to Direction!

Always check the direction of the vectors in the question. The path from Q to P is $\vec{QP}$, which is $-\boldsymbol{p}$. The path from S to P is $\vec{SP}$, which is $-\boldsymbol{s}$.

(a) To find $\vec{PR}$, we can follow the path $\vec{PS} + \vec{SR}$.

$$\vec{PR} = \vec{PS} + \vec{SR}$$

$$= \boldsymbol{s + 3p}$$

(b) To find $\vec{QS}$, we can follow the path $\vec{QP} + \vec{PS}$.

$$\vec{QS} = \vec{QP} + \vec{PS}$$

$$= \boldsymbol{-p + s} \quad \text{or} \quad \boldsymbol{s - p}$$

(c) To find $\vec{QR}$, we can follow the path $\vec{QP} + \vec{PS} + \vec{SR}$.

$$\vec{QR} = \vec{QP} + \vec{PS} + \vec{SR}$$

$$= -\boldsymbol{p} + \boldsymbol{s} + 3\boldsymbol{p}$$

$$= \boldsymbol{2p + s}$$

6)

(a) $\vec{OL} = \begin{pmatrix} \mathbf{1} \\ \mathbf{4} \end{pmatrix}$, $\vec{OM} = \begin{pmatrix} \mathbf{5} \\ \mathbf{1} \end{pmatrix}$, $\vec{ON} = \begin{pmatrix} \mathbf{-2} \\ \mathbf{0} \end{pmatrix}$

(b)

$$\vec{LM} = \vec{OM} - \vec{OL} = \begin{pmatrix} 5 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} \mathbf{4} \\ \mathbf{-3} \end{pmatrix}$$

(c)

$$\vec{MN} = \vec{ON} - \vec{OM} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} - \begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{-7} \\ \mathbf{-1} \end{pmatrix}$$

(d)

$$\vec{NL} = \vec{OL} - \vec{ON} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} - \begin{pmatrix} -2 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{3} \\ \mathbf{4} \end{pmatrix}$$

(e)

$$\vec{LM} + \vec{MN} + \vec{NL} = \begin{pmatrix} 4 \\ -3 \end{pmatrix} + \begin{pmatrix} -7 \\ -1 \end{pmatrix} + \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} 4 - 7 + 3 \\ -3 - 1 + 4 \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}$$

Deeper Insight: The Zero Vector and Closed Loops

The result $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is called the **zero vector**. This result is not a coincidence.

The journey $\vec{LM} + \vec{MN} + \vec{NL}$ describes a path that starts at point L and ends back at point L. Any vector journey that forms a closed loop will always have a resultant vector of zero.